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A New Loop Algebra and the Corresponding Computing Formula of Constant γ in the Quadratic-Form Identity, as Well as the Generalized Burgers Hierarchy

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Abstract A new loop algebra containing four arbitrary constants is presented, and the corresponding computing formula of constant γ in the quadratic-form identity is obtained in this paper, which can be reduced to computing formula of constant γ in the trace identity. As application, a new Liouville integrable hierarchy which possess bi-Hamiltonian structure and generalized Burgers hierarchy are derived.

Keywords Loop algebra · Computing formula of constant γ · Quadratic-form identity

1 Introduction

Loop algebra $\tilde{A}_1$ is used to constructing isospectral problem

$$\begin{cases} \psi_x = U\psi, & U, V \in \tilde{A}_1,\ \psi = (\psi_1, \psi_2)^T, \\ \psi_t = V\psi, & \lambda_t = 0, \end{cases} \tag{1}$$

whose compatibility condition exhibits a zero-curvature equation

$$U_t - V_x + [U, V] = 0 \tag{2}$$

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and the stationary zero-curvature equation

$$V_x = [U, V]. \tag{3}$$

Some hierarchies, such as AKNS hierarchy, KN hierarchy, BPT hierarchy [1] and so on [4–8], are derived with $\tilde{A}_1$ by establishing linear isospectral problem. In the following, we consider a general-form Lie algebra.

Set V_s to be an s-dimensional linear space with base $e_1, e_2, \dots, e_s$, and let

$$a = \sum_{i=1}^{s} a_i e_i = (a_1, a_2, \dots, a_s)^T,$$
$$b = \sum_{i=1}^{s} b_i e_i = (b_1, b_2, \dots, b_s)^T$$

be two elements of V_s and define a commutation operation as

$$[a, b] = \sum_{i,j=1}^{s} a_i b_j [e_i, e_j] = \sum_{i=1}^{s} c_i e_i = c = (c_1, c_2, \dots, c_s)^T. \tag{4}$$

It is easy to verify that V_s along with (4) becomes a Lie algebra.

A corresponding loop algebra $\tilde{V}_s$ is presented with base and commutation operation respectively as follows

$$\begin{aligned} &e_i(m) = e_i \lambda^m, \quad [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}, \\ &i, j = 1, 2, \dots, s, \ m, n = 0, \pm 1, \pm 2, \dots. \end{aligned} \tag{5}$$

The linear isospectral problem which is constructed by $\tilde{V}_s$ can be taken as

$$\begin{cases} \psi_x = [U, \psi], & U, V, \psi \in \tilde{V}_s, \\ \psi_t = [V, \psi], & \lambda_t = 0, \end{cases} \tag{6}$$

from which, the zero curvature equations (2) and (3) are easy to be derived, and we should define $\operatorname{rank}(U)$ and $\operatorname{rank}(V)$ and let the two arbitrary solutions V_1 and V_2 of (3) with the same rank have a linear relation

$$V_1 = \gamma V_2, \quad \gamma = \text{const}. \tag{7}$$

For $a, b \in \tilde{V}_s$, s-order matrix $R(b)$ is determined by

$$[a, b]^T = a^T R(b) \tag{8}$$

and constant matrix $F = (f_{ij})_{s\times s}$, is determined by

$$F = F^T, \quad R(b)F = -(R(b)F)^T. \tag{9}$$

We introduce quadratic-form identity functional

$$\{a, b\} = a^T F b, \quad a, b \in \tilde{V}_s \tag{10}$$

and consider functional

$$W = \{V, U_\lambda\} + \{\Lambda, V_x - [U, V]\},$$
$$U, V, \Lambda \in \tilde{V}_s \tag{11}$$

which has the following variation constraint conditions

$$\nabla_\Lambda W = V_x - [U, V] = 0, \tag{12}$$
$$\nabla_V W = U_\lambda - \Lambda_x + [U, \Lambda] = 0. \tag{13}$$

Then, we obtain

$$[\Lambda, V] - V_\lambda = \frac{\gamma}{\lambda} V, \quad \gamma = \text{const}. \tag{14}$$

Suppose the above conditions hold, the following formula holds:

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma} \frac{\partial}{\partial \lambda}\left(\lambda^\gamma \left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \quad 1 \le i \le l, \tag{15}$$

where γ is a constant to be determined. We call (15) the quadratic-form identity. It is worthwhile to mention that constant γ of (14) is the same as that of (15). In order to further discuss matrix $R(b)$ presented in the expression (9), we obtain the following results:

Definition Set V_s to be an s-dimensional linear space, and M_s a set of $s \times s$ matrices. Let R be a linear operator from V_s to M_s and satisfy

$$R(R^T(b)a) = [R(a), R(b)] = R(a)R(b) - R(b)R(a) \quad \forall a, b \in V_s, \tag{16}$$

then R is called a commuting operator on V_s. All the commutators on V_s constitute a set denoted by $K(V_s, M_s)$.

V_s is a Lie algebra with the commuting operation $[a, b]$ if and only if there exists $R \in K(V_s, M_s)$, so that

$$[a, b]^T = a^T R(b). \tag{17}$$

Let

$$b = (b_1, b_2, b_3)^T,$$
$$R(b) = \begin{pmatrix} \alpha_1 b_2 + \alpha_2 b_3 & \beta_1 b_2 + \beta_2 b_3 & \gamma_1 b_2 + \gamma_2 b_3 \\ -\alpha_1 b_1 + \alpha_3 b_3 & -\beta_1 b_1 + \beta_3 b_3 & -\gamma_1 b_1 + \gamma_3 b_3 \\ -\alpha_2 b_1 - \alpha_3 b_2 & -\beta_2 b_1 - \beta_3 b_2 & -\gamma_2 b_1 - \gamma_3 b_2 \end{pmatrix}, \tag{18}$$

then $R \in K(V_3, M_3)$ if and only if

$$\begin{cases} \alpha_2\gamma_1 - \alpha_1\gamma_2 = \beta_1\gamma_3 - \beta_3\gamma_1, \\ \alpha_2\beta_1 - \alpha_1\beta_2 = \beta_3\gamma_2 - \beta_2\gamma_3, \\ \alpha_3\beta_1 - \alpha_1\beta_3 = \alpha_2\gamma_3 - \alpha_3\gamma_2, \end{cases} \tag{19}$$

where $\alpha_i, \beta_j, \gamma_k$ are constants, $i, j, k = 1, 2, 3$.

Consider

$$R^*(b)=\begin{pmatrix} c_1b_2 & -c_2b_3 & c_3b_2\\ -c_1b_1-c_4b_3 & 0 & -c_3b_1+c_1b_3\\ c_4b_2 & c_2b_1 & -c_1b_2\end{pmatrix}, \tag{20}$$

where $c_1, c_2, c_3, c_4=\text{const.}$, $c_1^2+c_3c_4\neq 0$.

It is easy to verify that R^* satisfies (19) and $R^*\in K(V_3, M_3)$. Furthermore, the commuting operation of R^* is concise so that it is easy to be applied. The new loop algebra engendered by R^*, is denoted by $\tilde{V}_3^*$. And taking

$$F=\begin{pmatrix} -c_2c_3 & 0 & c_1c_2\\ 0 & c_1^2+c_3c_4 & 0\\ c_1c_2 & 0 & c_2c_4\end{pmatrix}, \tag{21}$$

we are easy to obtain the following results

$$F=F^T,\qquad R^*(b)F=-(R^*(b)F)^T \tag{22}$$

and the quadratic-form functional

$$\{a,b\}=a^TFb=-c_2c_3a_1b_1+c_1c_2a_3b_1+c_1c_2a_1b_2+c_2c_4a_3b_2+(c_1^2+c_3c_4)a_2b_2,$$
$$a,b\in\tilde{V}_3^*. \tag{23}$$

2 Computing Formula of Constant γ

In this section, in terms of loop algebra $\tilde{V}_3^*$, which brought from $R^*(b)$ (20), we obtain the corresponding computing formula of γ in the quadratic-form identity (15). Since constant γ in (15) is the same as that in (14), we start from (14) to solve it. For

$$U=e_0(\lambda)+\sum_{i=1}^{l}e_i(\lambda)u_i,\qquad \{e_0(\lambda), e_i(\lambda), 1\le i\le l\}\subset\tilde{A}_1,$$

we should define the proper rank number denoted by $\text{rank}(\lambda)$ and $\text{rank}(u_i)$ so that

$$\text{rank}(e_0(\lambda))=\text{rank}(e_i(\lambda)u_i)=\alpha\quad (1\le i\le l)$$

and simultaneously we call U the same rank, i.e. rank numbers of U is α, denoted by

$$\text{rank}(U)=\text{rank}\left(\frac{d}{dx}\right)=\alpha=\text{const.} \tag{24}$$

Suppose that solution of (3) is given by

$$V=\begin{pmatrix} v_1 & v_2\\ v_3 & -v_1\end{pmatrix},\quad v_k=\sum_{m\ge 0}v_{k,m}\lambda^{-m},\ k=1,2,3$$

and $\text{rank}(v_{k,m})$ is defined so that

$$\text{rank}(v_{k,m}\lambda^{-m})=\xi=\text{const.},\quad k=1,2,3,\ m\ge 0,$$

then V is called the same rank, denoted by

$$\mathrm{rank}(V)=\mathrm{rank}\left(\frac{d}{dt}\right)=\xi. \tag{25}$$

Lemma 1 *Suppose that conditions (24, 25, 7) and (12) hold, and*

$$V=\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}\in \tilde{V}_3^*,\qquad v_k=\sum_{m\geq 0} v_{k,m}\lambda^{-m},$$
$$G(V)=-c_2c_3v_1^2+2c_1c_2v_1v_3+c_2c_4v_2^2+(c_1^2+c_3c_4)v_2^2, \tag{26}$$

then the following equation holds

$$(G(V))_\lambda+\frac{2\gamma}{\lambda}G(V)=0, \tag{27}$$

where constant γ in (27) is the same as that in (15).

Proof From the conditions of Lemma 1, we are easy to test that (14) holds and unknown vector Λ $(\in\tilde{V}_3^*)$ exists, denoted by $\Lambda=(\eta_1,\eta_{2,}\eta_3)^T$. Turning (14) into equation systems, we read

$$c_1v_2\eta_1-(c_1v_1+c_4v_3)\eta_2+c_4v_2\eta_3=v_{1\lambda}+\frac{\gamma}{\lambda}v_1, \tag{28}$$

$$-c_2v_3\eta_1+c_2v_1\eta_3=v_{2\lambda}+\frac{\gamma}{\lambda}v_2, \tag{29}$$

$$c_3v_2\eta_1+(c_1v_3-c_3v_1)\eta_2-c_1v_2\eta_3=v_{3\lambda}+\frac{\gamma}{\lambda}v_3. \tag{30}$$

Let

$$(c_1v_3-c_3v_1)\times(28)+(c_1v_1+c_4v_3)\times(30),$$

then

$$v_2(c_1^2+c_3c_4)(v_3\eta_1-v_1\eta_3)=\frac{1}{2}(2c_1v_1v_3+c_4v_3^2-c_3v_1^2)_\lambda+\frac{\gamma}{\lambda}(2c_1v_1v_3+c_4v_3^2-c_3v_1^2). \tag{31}$$

Also let

$$-v_2(c_1^2+c_3c_4)\times(29),$$

then

$$c_2v_2(c_1^2+c_3c_4)(v_3\eta_1-v_1\eta_3)=-v_2(c_1^2+c_3c_4)\left(v_{2\lambda}+\frac{\gamma}{\lambda}v_2\right). \tag{32}$$

Comparing (31) with (32), we are easy to get the following results: (14) is solvable with respect to Λ if and only if

$$\frac{1}{2}(2c_1v_1v_3+c_4v_3^2-c_3v_1^2)_\lambda+\frac{\gamma}{\lambda}(2c_1v_1v_3+c_4v_3^2-c_3v_1^2)$$
$$-v_2(c_1^2+c_3c_4)\left(v_{2\lambda}+\frac{\gamma}{\lambda}v_2\right)$$

that is

$$(G(V))_\lambda + \frac{2\gamma}{\lambda} G(V) = 0. \qquad \square$$

Remark 1 If $G(V)$ of Lemma 1 is taken as

$$G(V) = \sum_{m\geq 0} g_m \lambda^{-m}, \tag{33}$$

then

$$g_m = \text{const.}, \quad m \geq 0.$$

We are easy to obtain $(G(V))_x = 0$ by turning $[U, V] = V_x$ into equation systems and using of transformation which is the same as that in the proof of the lemma. Hence, $g_{mx} = 0$, $g_m = \text{const}$.

Theorem 1 *Suppose that conditions* (24, 25, 7) *and* (12), *then computing formula of constant γ in the quadratic-form identity is obtained as follows*

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |G(V)| \quad (G(V) \neq 0). \tag{34}$$

Proof It is easy to see that $G(V)$ is a unitary function with respect to λ and independent of x. Also from Lemma 1, we know that (41) is convergence, that is $G(V) = g_{2\gamma}\lambda^{-2\gamma}$, therefore

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |G(V)| \quad (G(V) \neq 0). \tag{35}$$

$\square$

Remark 2 Theorem 1 is the main result in this paper, where $G(V)$ is the quadratic-expression of V. It is not difficulty to find the highest power λ^{-m_0} in the $G(V)$ so that $\gamma = \frac{m_0}{2}$.

3 Application

Taking the following linear isospectral problem

$$\begin{cases} \psi_x = [U, \psi], & U, V, \ \psi \in \tilde{V}_3^*, \\ \psi_t = [V, \psi], & \lambda_t = 0, \end{cases} \tag{36}$$

where

$$U = (q, r - \lambda, s)^T, \qquad V = (v_1, v_2, v_3)^T, \quad v_k = \sum_{m\geq 0} v_{k,m}\lambda^{-m},$$

$$\text{rank}(\lambda) = \text{rank}(q) = \text{rank}(r) = \text{rank}(\partial) = \text{rank}\left(\frac{d}{dx}\right) = 1. \tag{37}$$

Solving

$$V_x = [U, V] = (U^T R^*(V))^T = (R^*(V))^T U, \tag{38}$$

we give rise to

$$\begin{cases} v_{2,mx} = -qc_2v_{3,m} + sc_2v_{1,m}, \\ c_1v_{1,m+1} + c_4v_{3,m+1} = v_{1,mx} - qc_1v_{2,m} + rc_1v_{1,m} + rc_4v_{3,m} - sc_4v_{2,m}, \\ c_1v_{3,m+1} - c_3v_{1,m+1} = -v_{3,mx} + qc_3v_{2,m} + rc_1v_{3,m} - rc_3v_{1,m} - sc_1v_{2,m}, \\ v_{2,0} = \beta, \quad v_{1,0} = v_{3,0} = 0, \quad v_{2,1} = 0, \quad v_{1,1} = -\beta q, \quad v_{3,1} = -\beta s, \end{cases} \tag{39}$$

and

$$\begin{aligned} &\text{rank}(v_{k,m}) = m \quad (k = 1, 2, 3, m \geq 0), \\ &\text{rank}(V) = \text{rank}\left(\frac{d}{dt}\right) = 0. \end{aligned} \tag{40}$$

For arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ v_{2,m} \\ v_{3,m} \end{pmatrix} \lambda^{n-m}, \quad V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

then (38) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}], \tag{41}$$

the terms on the left-hand side of (41) are of degree ≥ 0, while the terms on the right-hand side of (41) are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_{1,n+1} \\ v_{2,n+1} \\ v_{3,n+1} \end{pmatrix}\right] = -\begin{pmatrix} 0 \\ c_1v_{1,n+1} + c_4v_{3,n+1} \\ c_3v_{1,n+1} - c_1v_{3,n+1}) \end{pmatrix}.$$

Denoting $V^{(n)} = V_+^{(n)} + \Delta_n$, $\Delta_n = (0, \delta_n, 0)^T \lambda^{-n}$, the following zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{42}$$

gives the new Lax integrable system

$$\begin{cases} q_{tn} = c_1v_{1,n+1} + c_4v_{3,n+1} - qc_11\delta_n - sc_4\delta_n, \\ r_t = \delta_{nx}, \\ s_t = -c_1v_{3,n+1} + c_3v_{1,n+1} - qc_3\delta_n + sc_1\delta_n. \end{cases} \tag{43}$$

Case 1 Taking $s = 0$, $\delta_n = \frac{c_3v_{1,n+1} - c_1v_{3,n+1}}{qc_3}$, hierarchy (43) was reduced to

$$\begin{cases} q_{tn} = \dfrac{c_1^2 + c_3c_4}{qc_2c_3} v_{2,n+1x}, \\ r_{tn} = \left(\dfrac{c_3v_{1,n+1} - c_1v_{3,n+1}}{qc_3}\right)_x, \end{cases} \tag{44}$$

$$\begin{aligned} u_t &= \begin{pmatrix} q \\ r \end{pmatrix}_{tn} = \begin{pmatrix} \frac{c_1^2+c_3c_4}{qc_2c_3} v_{2,n+1x} \\ (\frac{c_3v_{1,n+1}-c_1v_{3,n+1}}{qc_3})_x \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{qc_2c_3}\partial \\ \partial\frac{1}{qc_2c_3} & 0 \end{pmatrix} \begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix} = J \begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix}. \end{aligned} \tag{45}$$

Substituting $v_{2,0} = \beta = \text{const.} \neq 0$, $v_{2,0} = v_{1,0} = 0$ into (26), we have

$$G(V) = (c_1^2 + c_3c_4)\beta^2$$

and

$$\gamma = 0. \tag{46}$$

By making use of (15, 23) and (46), a direct calculation gives

$$\frac{\delta}{\delta u_i} - (c_1^2 + c_3c_4)v_2 = \frac{\partial}{\partial\lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\}, \quad u_1 = q, u_2 = r,$$

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} = -(c_1^2 + c_3c_4)v_2 = \frac{\partial}{\partial\lambda}\begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix}. \tag{47}$$

Comparing the coefficient of λ^{-n-1} in (47) leads to

$$\frac{\delta}{\delta u} - (c_1^2 + c_3c_4)v_{2,n+1} = -n\begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix}.$$

That is

$$\begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix} = \frac{\delta H_n}{\delta u},$$

where

$$H_n = \frac{c_1^2 + c_3c_4}{n}v_{2,n+1} \quad (n \geq 1, c_2 \neq 0, c_1^2 + c_3c_4 \neq 0). \tag{48}$$

Therefore, (45) can be written as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = J\frac{\delta H_{n+1}}{\delta u}. \tag{49}$$

From (39), an operator L meets

$$\begin{pmatrix} c_2(c_1v_{3,n+1} - c_3v_{1,n+1}) \\ (c_1^2 + c_3c_4)v_{2,n+1} \end{pmatrix} = L\begin{pmatrix} c_2(c_1v_{3,n} - c_3v_{1,n}) \\ (c_1^2 + c_3c_4)v_{2,n} \end{pmatrix},$$

where

$$L = \frac{1}{c_1^2 + c_3c_4}\begin{pmatrix} (c_1^2 + c_3c_4)r & qc_2c_3 + \partial\frac{1}{q} \\ (c_1^2 + c_3c_4)\partial^{-1}q\partial & -(c_1^2 + c_3c_4)\partial^{-1}r\partial \end{pmatrix}. \tag{50}$$

We observe that

$$JL = L^*J. \tag{51}$$

Hence, (45) can be written again as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = J\frac{\delta H_{n+1}}{\delta u} = JL^n\begin{pmatrix} c_2c_3\beta q \\ 0 \end{pmatrix} \quad (n \geq 0). \tag{52}$$

From (51), it is not difficulty to verify that Hamiltonian functions H_l $(l \geq 1)$ in (52) are involutive each other and each H_l $(l \geq 1)$ is the common conserved density of (52). Therefore, the hierarchy (52) is a Liouville integrable hierarchy. Furthermore, hierarchy (54) was reduced to Burgers hierarchy [2] when taking $c_1 = 0$, $c_2 = c_3 = c_4 = 1$. Hence, (52) is a new integrable hierarchy, we call it generalized Burgers hierarchy. Especially, when $n = 2$, hierarchy (52) was reduced

$$q_t = -\beta\left(rq_x + (qr)_x - \frac{q_{xx}}{q} - \frac{r_x}{q}\right),$$

$$q_t = -\beta\left(r^2 - \frac{1}{2}q^2 + \frac{q_{xx}}{q} + \frac{r_x}{q}\right)_x.$$

Let $q = 1$, $\beta = -1$, we obtained well-known equations

$$r_t = r_{xx} + 2rr_x.$$

Case 2 Taking $r = 0$, $\delta_n = 0$, hierarchy (43) was reduced to

$$\begin{cases} q_{tn} = c_1 v_{1,n+1} + c_4 v_{3,n+1}, \\ s_{tn} = -c_1 v_{3,n+1} + c_3 v_{1,n+1}, \end{cases} \tag{53}$$

$$\begin{aligned} u_t &= \begin{pmatrix} q \\ s \end{pmatrix}_{tn} = \begin{pmatrix} c_1 v_{1,n+1} + c_4 v_{3,n+1} \\ -c_1 v_{3,n+1} + c_3 v_{1,n+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{c_2} \\ -\frac{1}{c_2} & 0 \end{pmatrix} \begin{pmatrix} c_2(c_1 v_{3,n+1} - c_3 v_{1,n+1}) \\ c_2(c_4 v_{3,n+1} + c_1 v_{1,n+1}) \end{pmatrix} = J \begin{pmatrix} c_2(c_1 v_{3,n+1} - c_3 v_{1,n+1}) \\ c_2(c_4 v_{3,n+1} + c_1 v_{1,n+1}) \end{pmatrix}. \end{aligned} \tag{54}$$

Substituting $v_{2,0} = \beta = \text{const.} \neq 0$, $v_{2,0} = v_{1,0} = 0$ into (26), we have

$$G(V) = (c_1^2 + c_3 c_4)\beta^2$$

and

$$\gamma = 0. \tag{55}$$

By making use of (15, 23) and (55), a direct calculation gives

$$\frac{\delta}{\delta u_i} - (c_1^2 + c_3 c_4) v_2 = \frac{\partial}{\partial \lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\}, \quad u_1 = q, u_2 = r,$$

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta s} \end{pmatrix} - (c_1^2 + c_3 c_4) v_2 = \frac{\partial}{\partial \lambda} \begin{pmatrix} c_2(c_1 v_{3,n+1} - c_3 v_{1,n+1}) \\ c_2(c_4 v_{3,n+1} + c_1 v_{1,n+1}) \end{pmatrix}. \tag{56}$$

Comparing the coefficient of λ^{-n-1} in (56) leads to

$$\frac{\delta}{\delta u} - (c_1^2 + c_3 c_4) v_{2,n+1} = -n \begin{pmatrix} c_2(c_1 v_{3,n+1} - c_3 v_{1,n+1}) \\ c_2(c_4 v_{3,n+1} + c_1 v_{1,n+1}) \end{pmatrix}.$$

That is

$$\begin{pmatrix} c_2(c_1 v_{3,n+1} - c_3 v_{1,n+1}) \\ c_2(c_4 v_{3,n+1} + c_1 v_{1,n+1}) \end{pmatrix} = \frac{\delta H_n}{\delta u},$$

where

$$H_n = \frac{c_1^2 + c_3c_4}{nc_2} v_{2,n+1} \quad (n \geq 1, c_2 \neq 0, c_1^2 + c_3c_4 \neq 0). \tag{57}$$

Therefore, (54) can be written as

$$u_t = \begin{pmatrix} q \\ s \end{pmatrix}_t = J \frac{\delta H_{n+1}}{\delta u}. \tag{58}$$

From (39), an operator L meets

$$\begin{pmatrix} c_1 v_{3,n+1} - c_3 v_{1,n+1} \\ c_4 v_{3,n+1} + c_1 v_{1,n+1} \end{pmatrix} = L \begin{pmatrix} c_1 v_{3,n} - c_3 v_{1,n} \\ c_4 v_{3,n} + c_1 v_{1,n} \end{pmatrix},$$

where

$$L = \frac{1}{c_1^2 + c_3c_4} \times \begin{pmatrix} -c_1\partial - c_2(qc_3 - sc_1)\partial^{-1}(qc_1 + sc_4) & -c_3\partial - c_2(qc_3 - sc_1)\partial^{-1}(qc_3 - sc_1) \\ -c_4\partial + c_2(qc_1 + sc_4)\partial^{-1}(qc_1 + sc_4) & c_1\partial + c_2(qc_1 + sc_4)\partial^{-1}(qc_3 - sc_1) \end{pmatrix}. \tag{59}$$

We observe that

$$JL = L^*J. \tag{60}$$

Hence, (54) can be written again as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = J \frac{\delta H_{n+1}}{\delta u} = JL^n \begin{pmatrix} \beta(c_3q - c_1s) \\ -\beta(c_1q + c_4s) \end{pmatrix} \quad (n \geq 0). \tag{61}$$

From (60), it is not difficulty to verify that Hamiltonian functions H_l $(l \geq 1)$ in (61) are involutive each other and each H_l $(l \geq 1)$ is the common conserved density of (61). Therefore, the hierarchy (61) is a Liouville integrable hierarchy. When taking $c_1 = 0$, $c - 2 = -1$, $c_3 = c_4 = 1$. Furthermore, hierarchy (61) can be reads

$$\begin{aligned} u_t &= \begin{pmatrix} q \\ r \end{pmatrix}_t = J \frac{\delta H_n}{\delta u} = JK \frac{\delta H_{n-1}}{\delta u}, \\ J &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} q\partial^{-1}s & q\partial^{-1}q - \partial \\ s\partial^{-1}s - \partial & s\partial^{-1}q \end{pmatrix}, \\ K &= JL = \begin{pmatrix} s\partial^{-1}s - \partial & s\partial^{-1}q \\ -q\partial^{-1}s & -q\partial^{-1}q + \partial \end{pmatrix}, \end{aligned} \tag{62}$$

K, J are Hamiltonian operator, $\alpha J + \gamma K$ (α, γ are constants) is also Hamiltonian operator, so hierarchy (62) has bi-Hamiltonian structure.

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